

DETERMINISTIC NUMERICAL SIMULATION OF A NON-ISOTHERMAL BLACKBODY EFFECTIVE EMISSIVITY USING A COUPLED FREDHOLM INTEGRAL EQUATION SYSTEM

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ABSTRACT
Radiation thermometry involves the modelling of radiative heat transfer interactions between thermal radiation sources such as blackbodies and thermal radiation detectors such as pyrometers in order to deduce temperatures. When furnaces and heat-baths are modelled as blackbodies the corresponding effective emissivity becomes essential in order to adequately characterise non-ideal systems. Due to the complexity that is involved in such systems analytical formulae are generally only mathematically tractable for isothermal systems and simple geometries. Systems exhibiting more complex geometries and optical/thermal characteristics are then frequently only amenable to further analysis using stochastic based simulation techniques. One of the problems associated with stochastic techniques is that they suffer from a lack of independent verification and validation bench marks for non-isothermal conditions. In this paper we adopt a physical based modelling approach in order to independently determine the effective emissivity of a high temperature blackbody that exhibits a non-isothermal heat distribution by using a deterministic numerical simulation. We model the system in terms of a generalized Fredholm integral equation system and develop a numerical solution approach using a modified collocation method that may incorporate the zonal technique, and briefly contrast the merits of our approach with that of multi-parametric homotopy techniques that are based on the Adomian decomposition method.

KEY WORDS
Nonisothermal blackbody, effective emissivity, Fredholm integral equation, radiative heat transfer, physical based modelling.

1 Introduction
The high accuracy measurement of physical experimental data is conventionally specified in terms of the International System of Units (Le Système International d’Unités, SI) which provides a modern form of the metre-kilogram-second (MKS) based metric system that was originally adopted in 1960 by the international scientific community. Based on various scientific and technological developments over the last two decades as earlier reported by Milton et al [24] it now anticipated by White & Fischer [38] that the time for revisions to the SI seven base units may be introduced. The envisioned changes will occur by defining the base units of measurement in terms of the fundamental physical constants of nature. Since these improvements to the SI may be officially introduced to the international scientific community possibly as early as by 2018 at the 26th General Conference on Weights and Measures (CGPM) it is incumbent upon mathematicians, physicists and engineers working in research work that is reliant on very high accuracy measurements that will be impacted upon by these changes to understand and incorporate the newer developments.

Amongst various improvements in the fundamental physical constants of nature that have been achieved and reported in the literature it has been proposed that by fixing the value of the Boltzmann constant $k_B$ that the kelvin would no longer be specified in terms of the thermodynamic state of the triple point of water. If this redefinition of the SI temperature scale occurs then changes in the envisioned temperature accuracies achievable through the new SI route for traceability will mainly occur for temperatures below 20 K and above 1300 K.

In current experimental measurement research work improved low temperatures are investigated mainly using acoustic gas thermometry techniques as discussed by Moldover et al [25] whilst high temperatures are investigated using either radiation thermometry or optical radiometry based techniques above the silver fixed point which occurs at a temperature fixed by definition as $T_{Ag} = 1234.93$ K. The silver point in the existing International Temperature Scale of 1990 (ITS-90) currently delineates the separation of contact thermometry and radiation thermometry achievable scale realizations. As a result due to the envisioned changes in the temperature scale definitions there will be an increased demand by users in the research and industry sectors to adequately characterise blackbody thermal radiation sources which are used as reference standards in temperature laboratories to calibrate instruments such as pyrometers, radiometers, thermal imaging systems and heat flux sensors. For blackbody thermal radiation sources the corresponding effective emissivity $\varepsilon_e$ is considered essential in order adequately characterise non-ideal systems such as non-isothermal blackbodies in order to uti-
lize the blackbody for high accuracy experimental research work.

Alternatives to closed form analytical formulae for these simplifying special cases are generally limited to stochastic based techniques such as optical ray tracing schemes and Monte Carlo simulations for more complex geometries and boundary conditions. In this paper we will mathematically model a non-isothermal high temperature blackbody’s effective emissivity using a deterministic approach directly in terms of a coupled system of generalized Fredholm integral equations, in order to provide a numerical simulation scheme that may be used in verification and validation (V&V) benchmark simulation results.

2 Mathematical Modelling

The mathematical modelling of blackbody emissivities for modern scientific metrology studies may be traced back to a set of papers by Bedford & Ma [10]. In their investigations it was determined that the integrated cavity emissivities of non-isothermal cavities are very strongly dependent on the wavelength $\lambda$. For the particular cases they considered it was determined that for normal spectral emissivities $\varepsilon_n$ the variation ranged from 4% to 6% and that in the case of a hemispherical spectral emissivity $\varepsilon_h$ there was a variation from 17% to 20% when a 1% representative temperature variation in the blackbody’s surface temperature at 1300 K was present for wavelengths from 0.3 $\mu$m to 1.0 $\mu$m.

In many modern research laboratories radiation thermometers are usually operated at wavelengths from 400 nm to 1050 nm for high temperature blackbodies above 600 °C with Si based photo-detectors. This choice of operating wavelength implicitly minimizes blackbody emissivity uncertainties, however many radiation thermometers particularly in industrial laboratories for mid-range temperatures from 100 °C to 600 °C utilize InGaAs or Ge based photo-detectors amongst other choices which operate over wavelengths from 1 $\mu$m to 3 $\mu$m, and for low-range temperatures from 0 °C to 100 °C utilize wide-band detectors over wavelengths from 3 $\mu$m to 15 $\mu$m using a mixture of options such as bolometers, pyro-electric and thermo-piles for the detectors as discussed in Zhang & Machin [39]. Due to these practical considerations in the use of thermal radiation detectors the presence of non-isothermal conditions in blackbodies remains a measurement issue.

Our focus in this paper is to mathematically model and develop a numerical approach to investigate effective emissivities for a high temperature blackbody with non-isothermal temperature distribution similar to that of a Thermo-Gauge blackbody as illustrated in Figure 1. This type of cylindrical blackbody of nominal total length $L = 300$ mm and diameter $D = 25.4$ mm is usually operated at a nominal power rating of 48 kW and is currently used by many international laboratories as a high accuracy temperature standard for temperatures from about 550 °C up to approximately 3000 °C which is the upper physical endurance limit of graphite which is the material used to construct the blackbody surface.

This particular type of blackbody was previously experimentally investigated by Kozlova et al [19] using a LP3 (“Linear-pyrometer 3”) which uses an advanced optical focusing system that allows for the pyrometer to physically focus on and measure small optical spot sizes over the blackbody surface temperature distribution in longitudinal and axial directions. For our purposes we will assume that the physical measurements reported by Kozlova et al are sufficiently accurate and representative for longitudinal/radial temperature profiles in order to investigate the effective emissivity of a blackbody operating at a nominal temperature of 1085 °C i.e. at 1358.15 K although we comment that in practise for a temperature range $1000 \leq t/[^{\circ}\text{C}] \leq 3000$ the practical expanded temperature uncertainties will typically lie in the range $0.3 \leq u(t)/[^{\circ}\text{C}] \leq 1.8$ as discussed by Kozlova et al.

Earlier effective emissivity studies by Bedford & Ma for isothermal/non-isothermal cavities incorporated refinements such as the zonal approximation for analytical calculations which exhibited discontinuities such as the angle factors in order to simplify the calculations and are conveniently summarized in a later paper by Ohwada [28]. Building on this work later approaches incorporated stochastic based techniques that implemented Monte Carlo based simulation techniques first for isothermal blackbody conditions and which were then later extended for nonisothermal blackbody conditions as reported by Sapritsky & Prokhorov [33, 34]. Whilst more recent stochastic based approaches tend to rely on Monte Carlo simulations the verification and validation (V&V) of these codes as discussed in Lucas [21] still tend to be tested and benchmarked against earlier results such as that of Bedford & Ma [9] particularly for blackbodies with cones/cylinders and axial symmetries. More recent Monte Carlo studies have been undertaken by Prokhorov & Hannsen [29, 30] for isothermal/non-isothermal blackbodies with inclined bottoms which introduce some specialist functionality for radiation thermometry experiments but one of the key challenges that remain in stochastic based simulations is that of an independent V&V of the research and commercial computer codes that are used.

From a recently reported discussion by Liu et al [20] it has been established that Monte Carlo simulations are

![Figure 1. Cylindrical high temperature Thermo-Gauge blackbody (Graphic Source: Hartmann et al [16])](image-url)
presently the only known modern approach to calculate integrated effective emissivities for non-isothermal non-axisymmetric cavities i.e. blackbody surfaces that are not directly amenable to analytical techniques. This observation is due to the fact that the existing approach developed by Liu et al involves the use of the commercial software package Ansys Fluent to determine the view factors in a blackbody geometry. The weakness of the application of Ansys Fluent to blackbody simulations is that it suffers from the limitation that the emissivity of a non-isothermal cavity which is known to vary with the wavelength $\lambda$ should have the radiant flux terms calculated with the full Planck equation however Ansys Fluent is more amenable to calculating radiant flux terms with the Stefan-Boltzmann based equation $E = \sigma T^4$ which corresponds to total radiation emissions. Whilst Liu et al modified their approach to calculate the radiation transport for specific wavelengths over a particular wavelength spectrum by using an equivalent temperature $T_e$ from the transformation equation $\sigma T^4_e = [c_1 \lambda^{-5}] / [\exp(c_2/\lambda T) - 1]$ this approach is less than ideal particularly for scientific metrology work since the effective temperature $T_e$ is not a physical quantity that can be independently experimentally tested. Due to these reasons a deterministic approach to calculating an effective emissivity for a non-isothermal blackbody may thus provide a convenient and valuable independent verification and validation (V&V) approach for testing and benchmarking existing stochastic based simulation codes techniques and is the motivation for the research work addressed in this paper. Our main focus in this paper will be to mathematically formulate and develop a numerical technique to directly solve the system of integral equations for diffuse cavities as discussed by Prokhorov et al [31]. Although in the original work by Bedford & Ma which formulated the spectral directional effective emissivity in terms of the of the spectral exittance $M_2$ more recent analysis by Prokhorov et al has clarified that the use of spectral existances can only be formally employed for emissivity definitions in cases where the blackbody internal walls exhibits Lambertian behaviour i.e. the blackbody has diffusely emitting and reflecting surfaces. In practical terms this specification is not too onerous since earlier work by Bedford & Ma compared theoretical predictions between purely diffuse, purely specular, and variations between diffuse/specular surface respectively with experimental irradiance measurements by Schornhorst & Viskanta [35] and concluded that the diffuse model is by and large a good approximation. Nevertheless in the interest of physical accuracy we will employ spectral radiances in units of $[\text{W m}^{-3} \text{sr}^{-1}]$ so that the effective emissivity is formally defined as $\varepsilon_e(\lambda, \xi, \omega, T_0) = \frac{L_{\lambda,bb}(\lambda, \xi, \omega, T_0)}{L_{\lambda,bb}(\lambda, T_0)}$ and $L_{\lambda,bb}(\lambda, T_0) = c_1 \{ \pi \lambda^5 \left[ \exp \left( \frac{c_2}{\lambda T_0} \right) - 1 \right] \}^{-1}$ where $L_\lambda = \frac{\partial L}{\partial \lambda}$ and $L_{\lambda,bb}$ are the spectral radiances of the actual blackbody and idealized blackbody, $c_1$ and $c_2$ are the first and second radiation constants, $\lambda$ is the wavelength, $\xi$ is a position vector, $\omega$ is an orientation i.e. direction vector, and $T_0$ is a reference temperature respectively. By using the above fundamental definition of emissivity various specific types of emissivity may be defined such as band-limited and total integrated effective emissivities, and in particular the spectral effective emissivity for non-isothermal and isothermal cavities may be defined as

$$
\varepsilon_e(\lambda, \xi, T_0) = \varepsilon(\lambda) \frac{\exp \left( \frac{c_2}{\lambda T_0} \right) - 1}{\exp \left( \frac{c_2}{\lambda T} \right) - 1} + [1 - \varepsilon(\lambda)] \int_{A'} \varepsilon_e(\lambda, \xi', T_0) K(\xi, \xi') \, dA' \tag{1}
$$

$$
\varepsilon_e(\lambda, \xi) = \varepsilon(\lambda) + [1 - \varepsilon(\lambda)] \int_{A'} \varepsilon_e(\lambda, \xi') K(\xi, \xi') \, dA' \tag{2}
$$

$$
K(\xi, \xi') = \frac{dF_{d\lambda\rightarrow dA'}(\xi, \xi')}{dA'} \tag{3}
$$

and $dF_{d\lambda\rightarrow dA'}(\xi, \xi')$ is the view factor between the elemental areas $dA(\xi)$ and $dA'(\xi')$. The above system of integral equations formally define the local directional effective emissivity since this is equal to the local hemispherical effective emissivity in the special case of a blackbody cavity with perfectly diffuse walls. Since the method of integral equations yields the distributions of the local effective emissivity over the internal surfaces of the blackbody in practical situations one must also consider the effect of vignetting i.e. situations where the entire blackbody surface is visible to a detector and those where only portions of the internal blackbody surface are visible.

Different approaches to incorporate the vignetting effect have been discussed in the literature but for our purposes we adopt that of De Lucas such that in his notation the integrated effective emissivity $\varepsilon^*_a$ is of the form

$$
\varepsilon^*_a(\lambda, T_a, A_d) = \frac{\iint \varepsilon_a(\lambda, 2\pi, T, T_a, x, y) \, dA(x, y) \, dF(x, y, A_d)}{\iint dA(x, y) \, dF(x, y, A_d)} \tag{4}
$$

Figure 2. Illustration of blackbody geometry definitions
in the most general case. In the above equation $A_d$ is the area of the detector, $dA(x,y)$ is an elemental area on the blackbody surface, $dF(x,y,A_d)$ is the view factor between the elemental area $dA$ and the detector’s area $A_d$, $\varepsilon_a$ is the hemispherical effective emissivity that results from the solution of the integral equations, $T'(x,y)$ is the blackbody temperature and $T_a$ is a reference temperature which is generally taken as the temperature at the bottom/center of the blackbody. The above system of equations completely specifies the mathematical modelling of the effective emissivity of a non-isothermal blackbody and our remaining objective is to formulate the numerical solution approach in order to solve the underlying system of integral equations. As per the discussion of De Lucas & Segovia [22] who recently reported on research work they conducted on characterising the effective emissivity of a heat-pipe based isothermal cylinder/cone blackbody with a lid and perfectly diffuse walls using a Monte Carlo based photon trajectory scheme, we will also adopt their approach to use an internal consistency approach for validation as previously also undertaken by De Lucas in other stochastic simulations. This validation approach which at a conceptual level is applicable to both stochastic as well as deterministic modelling approaches essentially involves the use of different numerical techniques to independently solve the same underlying mathematical model since different results reported in the literature do not always consistently account for and adequately address the vignetting effect in blackbodies with lids thus making direct comparisons from the literature for different laboratories with various blackbodies configurations and different temperature gradient information difficult. In our case there is the additional complication of a non-isothermal blackbody surface so we will compare our solution of the Fredholm integral equation system with the accepted classical approach of an earlier paper by Ohwada [27] who used the the previously discussed Bedford & Ma approach with a radiative heat transfer zonal approximation and a series technique to determine the blackbody effective emissivity. In the most general case the previous non-isothermal and isothermal integral equations may be reformulated as a Fredholm integral equation of the second kind which determines the mathematical characteristics of the blackbody effective emissivity system of the form

\[ \Lambda(\xi) = \varepsilon_a(\xi) - \rho \int_a^b K(\xi,\xi') \varepsilon_a(\xi') \, d\xi' \]

(5)

\[ \Lambda(\xi) = \begin{cases} 
\rho \exp \left[ -\frac{(\xi)}{\tau_a} \right] & \text{for isothermal emissivity} \\
\exp \left[ -\frac{(\xi)}{\tau_a} \right] & \text{for spectral emissivity} \\
\rho \exp \left[ -\frac{(\xi)}{\tau_a} \right] & \text{for total emissivity}
\end{cases} \]

(6)

\[ \rho = 1 - \varepsilon_a, \quad a \leq \xi, \xi' \leq b \]

(7)

Some further analysis is necessary in order to understand the notation of the generalized Fredholm equation by observing that the non-isothermal spectral emissivity model is in terms of a system of multi-variate integral equations $\varepsilon_s = \int_a^b e_s(x,y) \, d\xi$ for $i = 1,\ldots,\infty$ that are defined for all points $\xi \in A$ on the blackbody surface $A$ by using the definition $K = \frac{dF(x,y)}{dA}$. In the effective emissivity equation the surface integral of the kernel is evaluated by considering all the corresponding points $\xi'$ for the elemental area $dA'$ that lie on the surface for a particular point $\xi$ for the elemental area $dA$ on the surface where $d$ is the distance between $dA$ and $dA'$. When solving the above Fredholm integral equation system the main challenge which will arise is that of singularities in the kernels. In for example conical blackbodies at the apex of the cone the view factor is technically indeterminate although it can in principal be addressed through the application of L'Hôpital’s rule. Unfortunately this approach is not viable in for example cylindrical blackbodies on the intersection between the bottom and side walls where the view factor is technically indefinite and as a result discontinuities in the kernels can present considerable challenges in practical effective emissivity studies. Issues of view factor singularities have traditionally been addressed through the use of the zonal approximation as originally developed and applied by Bedford & Ma which approximates $\varepsilon_a(x)$ as a slowly varying function of $x$ of form $\int_a^b \varepsilon_a(\xi) \, dF_\theta(\xi) \approx \int_a^b \varepsilon_a(\xi) \, dF_\theta(\xi) \times \int_{\xi_1}^{\xi_2} dF_\theta(\xi)$. An alternative to the zonal approximation is through the use of generalized quadrature formulae as later developed by Chandos & Chandos [13] who used an approach undertaken by Atkinson [2] such that $\int_a^b \varepsilon_a(\xi) \, dF_\theta(\xi) \approx \sum_{i=1}^n \left( \alpha_i \varepsilon_a(\xi_{i-1}) + \beta_i \varepsilon_a(\xi_i) \right)$ where $\alpha_j = \frac{1}{\xi_j-\xi_{i-1}} \int_{\xi_{i-1}}^{\xi_j} (\xi - \xi_{i-1}) F(\xi,x) \, d\xi$ and $\beta_j = \frac{1}{\xi_j-\xi_{i-1}} \int_{\xi_{i-1}}^{\xi_j} (\xi - \xi_i) F(\xi,x) \, d\xi$.

As per the overview by Prokhorov et al the method of iterations, the series method, and the quadrature method all in principal allow for an exact solution of the mathematical model. In the series method a trial solution of the form $\varepsilon_a(\xi) = \Lambda(\xi) + \sum_{k=1}^\infty \rho^k E_k(\xi)$ is assumed for the functions $E_k(\xi)$ such that a Neumann series results for the effective emissivity of the form $\varepsilon_a(\xi) = \Lambda(\xi) + \sum_{k=1}^\infty \rho^k K_\nu(\xi,\xi') \Lambda(\xi')$ $d\xi'$ where $K_\nu(\xi,\xi') = K(\xi,\xi')$ if $i = 1$, and $K_\nu(\xi,\xi') = \int_a^b K(\xi,\xi')K_\nu(\xi,-1)(\xi',\xi_\nu) \, d\xi'$ if $i > 2$.

Since the line integral evaluation approach to determine the integration limits $a$ and $b$ is rather cumbersome the multi-dimensional form for the Fredholm integral equation of form $f(x_0) = \int_a^b K(x_0,x_1) \, dx_1 \, g(x_0)$ where $g : E \rightarrow \Re$ and $K : E \times E \rightarrow \Re$ are known and $f : E \rightarrow \Re$ is unknown may be used instead in order to avoid sequential transformation mappings of the integral equation to a standard form. When a direct evaluation of the surface integral is desired the von Neumann series solution is known to take the exact mathematical form $f(x_0) = f_0(x_0) + \sum_{i=1}^\infty f_i(x_0,x_n) \, dx_1 \, f_0(x_0) = g(x_0)$ and $f_n(x_0,x_n) = g(x_n) \prod_{k=1}^{n-1} K(x_{k-1},x_k)$ as discussed by Doucet et
al [14]. The above multi-dimensional von Neumann series solution is technically an exact deterministic mathematical solution but is however extremely difficult to computationally implement for practical problems using traditional mathematical techniques unless approached using statistical based Monte Carlo sampling techniques.

A survey of numerical methods for solving integral equations is reported by Atkinson [3] who investigated techniques for the general integral equation of form \( x = \mathcal{K}(x) \) where \( x \) is a point in a space \( \mathcal{X} \) and \( \mathcal{K} \) is an operator acting on \( x \). In general the two main techniques for solving integral equations are that of projection methods which encompass collocation Galerkin techniques as discussed by Atkinson & Potra [5, 6], and that of Nyström based methods. In both cases the original equation \( x = \mathcal{K}(x) \) is replaced by a sequence of finite dimensional approximations \( x_n = \mathcal{K}(x_n) \) where \( n \) is some suitable discretization parameter and the system is investigated for convergence as \( n \to \infty \). The main issue with standard techniques when applied to radiative heat transfer equations is that they cannot be conveniently reformulated using continuous operators, and as a result modifications to the kernel such as the application of the zonal approximation are necessary in order to apply standard solution techniques.

We comment that the generalized Fredholm equation when it is converted to a corresponding line integral after a suitable parametrization is of the form \( y(x) = f(x) + \int_{\Omega} \mathcal{K}(x,t,y(t)) dt \), \( a \leq x \leq b \), \( \Omega = (a,b) \) which is an integral equation of the Urysohn type. This type integral equation may be solved using the Newton-Kantorovich method as reported by Saberi-Nadjafi & Heidari [32] if the limits of integration \( a \) and \( b \) are known. A potential alternative if the integral is not formulated as a line integral after suitable transformations is to directly compute the surface integral using a radial basis function (RBF) approach as discussed by Assari et al [1] however a potential limitation with this approach is in terms of the existence of a corresponding inverse Kansa matrix.

Due to the above combination of complicating issues in this paper we will consider an approximate discretization using a finite volume type methodology which may be considered as a type of modified collocation technique.

3 Numerical Simulation Techniques

Previous investigations used view factors that were calculated in terms of analytical formulae since specific geometries such as cones, cylinders, and oblique sections were considered in which analytical formulae were available. In this paper we will however consider a direct numerical approximation for arbitrary surfaces \( S_i \) and \( S_j \). As per an analysis performed by Vujčić et al [37] who investigated radiative heat transfer simulations where the Monte Carlo method was coupled with the finite element technique it was concluded that additional numerical resources would be required for individual finite element view factors when finer mesh discretization occurred. In their investigations Vujčić et al concluded that using finer meshes unthinkingly without adequately accounting for increased computations in the Monte Carlo based individual element view factor calculations could lead to an unexpected decrease in overall simulation accuracies. Based on these observations Bopche & Sridharan [11] considered the application of Stoke’s theorem for various concentric/non-concentric configurations and concluded that the historical contour integral technique as originally developed by Sparrow [36] for two finite areas \( A_1 \) and \( A_2 \) of the form

\[
F_{12} = \frac{1}{2\pi} \int_{\Gamma_1} \int_{\Gamma_2} \ln(S) \, d\Gamma_1 \cdot d\Gamma_2
\]

can still present a simple, effective and accurate tool for three dimensional view factor evaluations. As a result in this paper we will employ a modern update on the contour integration approach in order to determine view factors.

One particular contour integration is that developed by Mazumder & Ravishankar [23] who approximate the view factor between a surface \( P \) and \( Q \) as which are arbitrary planar polygons where \( P \) has \( M \) vertices and \( Q \) has \( N \) vertices. Mazumder & Ravishankar report that their approach yields good results if a 10-point Gauss–Legendre quadrature is used to compute the definite integral terms and we comment that this approach for view factors is recommended in the event when complex/arbirtary shapes are present.

In our particular case since the blackbody has a relatively simple shape but a more complex temperature distribution we instead opt for a more simple calculation as proposed by Francisco et al [15] that directly applies Stoke’s theorem to directly estimate the view factor \( F_{ij} = \frac{1}{4\pi} \int_{\Gamma_j} \int_{\Gamma_i} \frac{(\cos \theta_i)(\cos \theta_j)}{\pi d_{ij}} \, dA_j \, dA_i \) for flat polygons as

\[
F_{ij} \approx \frac{1}{2\pi d_{ij}} \sum_{k=1}^{nk} \sum_{l=1}^{nl} \ln(\delta_{kl}) S_{ik} S_{lj} \delta_{ij} \text{ dist}(r_k, r_l)
\]

where area \( A_i \) has a contour \( C_i = \bigcup_{k=1}^{nk} r_{ik} \), area \( A_j \) has a contour \( C_j = \bigcup_{l=1}^{nl} r_{lj} \) and \( \delta_{ij} = \text{dist}(r_k, r_l) \) is the distance between points on the respective contours. We comment that since we only consider flat polygon elements on the blackbody’s surface that it follows that \( F(S_i, S_j) = 0 \) since the element surface is convex, and that the view factor algorithm as developed by Francisco et al offers a quick and preferred approach to determine view factors when simple/regular shapes are present such as triangles and linear/curvilinear quadrilaterals.

To perform a numerical simulation we let the blackbody surface \( A \) have \( N \) sub-surfaces so that \( S = \bigcup_{k=1}^{N} S_k \) where each surface \( S_k \) is composed of a flat polygon. For simplicity we limit the polygons to either triangles or quadrilaterals and for which formulae to calculate the respective areas \( A_k \) and centroids \( c_k \) are well known. In the event that the surface consist of arbitrary flat polygons specified by vertices \( \{ [x_0, y_0], \ldots, [x_{n-1}, y_{n-1}] \} \) such that the last vertex \( (x_n, y_n) \) is coincident with the first vertex \( (x_1, y_1) \) then the area \( A \) and centroid \( c = [c_x, c_y]^T \) may be calculated as \( A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - y_i x_{i+1}) \), \( c_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(y_{i+1} - y_i) \) and \( c_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_{i+1} - x_i) \) as discussed by Bourke [12].

Making the assumption of a constant material emissivity \( \epsilon \) the corresponding system of equations for the un-
known effective emissivities $\mathbf{e} = [\varepsilon_1, \ldots, \varepsilon_N]^T$ is then

\begin{align}
Ae &= B \\
A_{ik} &= \begin{cases} 
\rho F_{ik}, & i \neq k \\
-1, & i = k
\end{cases} \\
B_i &= (\rho - 1) \frac{e^{\alpha_i/|z|} - 1}{e^{\alpha_i/|z|} - 1}
\end{align}

The view factor $F_{ik}$ may be calculated either with a contour integration approach as previously discussed, or alternately it may be directly approximated as

\begin{align}
F_{ik} &= \frac{\cos \theta_i \cos \theta_k A_{ik}}{\pi d_{ik}^2} \\
&= \frac{-(\mathbf{n}_i \cdot \mathbf{d}_{ik})(\mathbf{n}_k \cdot \mathbf{d}_{ik})A_{ik}}{\pi ||\mathbf{d}_{ik}||^4}
\end{align}

since from standard vector geometry $\mathbf{d}_{ik} = -\mathbf{d}_{ik}$.

Our approach thus differs from the more conventional approach as discussed by Atkinson [4] for multi-variable integral equations where for a given planar region $\mathcal{R} = [a, b] \times [c, d]$ the unknown function is typically approximated with Lagrangian polynomials such as for example a bilinear approximation of the form $g(x, y) \approx \frac{(b-x)(d-y)}{(b-a)(d-c)} g(a, c) + \frac{(b-x)(y-c)}{(b-a)(d-c)} g(a, d) + \frac{(x-a)(d-y)}{(b-a)(d-c)} g(b, c) + \frac{(x-a)(y-c)}{(b-a)(d-c)} g(b, d)$ for quadrilaterals and similar formulations for triangulations. In our approach we assume that the effective emissivity is approximately constant for a particular area element and refine the solution through an increasing number of elements. The benefits are therefore a considerably reduction in the computational complexity, and consistency with the later approaches adopted by Bedford & Ma who used the zonal approximation. We comment that if a conventional discrete collocation were implemented then the system would become ill-defined due to the presence of singularities.

Considering the surface temperature distribution we opt to utilize representative experimental temperature data for a nominal blackbody temperature of 1085 °C i.e. $T_0 = 1358.2$ K where we model the longitudinal temperature drop along the length of the cylindrical blackbody as $\Delta T(z)/|z| = \sum_{i=\epsilon}^{\lambda} \alpha_i (x_i Z \times 3^i)$. In our case for the previously mentioned Thermo-Gauge blackbody the coefficient values for the representative longitudinal temperature that we consider are $\alpha_4 = -1.91916701480 \times 10^{-6}$, $\alpha_3 = 2.155875374087 \times 10^{-4}$, $\alpha_2 = -1.76573892447 \times 10^{-2}$, $\alpha_1 = -1.237305270116 \times 10^{-1}$, and $\alpha_0 = 2.569151012969 \times 10^{-1}$ respectively where the longitudinal temperature difference is defined as $\Delta T(z) = T(z) - T_0$. The radial temperature drop is then modelled for convenience as an asymmetric Gaussian centred on a spot at the back of the base surface of the blackbody such that

$\Delta T(R, \theta)/|K| = -A_o \left\{ 1 - \exp \left[-\left(\frac{R^2 - y_0^2}{2 \alpha^2}ight)\right]\right\}$

located at $x_o/[m] = -4 \times 10^{-3}$, $y_o/[m] = 2 \times 10^{-3}$, $A_o/[K] = 6 \times 10^3$, $\sigma_o/[m] = 1$, and $\sigma_f/[m] = 1$ respectively where the radial temperature difference is defined as $\Delta T(R, \theta) = T(R, \theta) - T_0$. Under these modelling assumptions the radial temperature drop on the base of the blackbody is roughly $\Delta T(R, \theta) = -2.3^\circ C$ and the axial temperature drop along the length of the blackbody is roughly $\Delta T(z) = -78^\circ C$ respectively which is consistent with the typical temperature variations experienced in a physical Thermogauge blackbody when operated at $T = 1358.2$ K.

In practical terms it is seldom necessary to simulate for the entire length of the cylinder wall as discussed by Bedford [8] since the main sources of irradiation incident on the pyrometer’s field of view is from the base of the cylinder and the walls near the base of the cylinder. As a result it is usually common practise to simulate only for a region of the blackbody that is visible within a pyrometer’s field of view for a proportion $L/D$ and to calculate the effective emissivity as a limit as $(L/D) \rightarrow (L_{\text{max}}/D)$. For our particular case of a Thermo-gauge cylindrical blackbody which is constructed out of graphite the material emissivity may be estimated as $\varepsilon = 0.855$ as per the data reported by Neuer [26] and Kostanovskii et al [18]. Using this data with $\lambda = 900$ nm, $T_0 = 1358.2$ K, $L/D = 1.9685$, and $D = 25.4$ mm with the geometrical mesh as illustrated in Figure 3 then produces the final emissivity distribution results as shown in Figure 4.

4 Conclusions

In a previous work Babolian & Mordad [7] reported that they successfully used hat functions to solve, amongst other classes of integral equations, a system of Fredholm integral equations which converted the system of integral equations into a corresponding non-linear system which they approximately solved used a least squares technique for over-determined systems. This outcome suggests that it may be possible to indirectly apply the well known hat basis function approximations that are common in the finite element
method (FEM) as discussed by Zienkiewicz et al [40] in order to refine the integral terms \( \int_a^b \varepsilon_s(\xi_i)K(\xi_i, \xi_j)\,dA' \) in the constituent defining equations. The main conceptual difficulty with this approach is that the resulting system apart from being non-linear will in addition be substantially more complicated and it is not clear if this approach will offer substantial accuracy rewards as opposed to a more conventional mesh refinement approach.

Alternatives to a hat function approach include the well known homotopy analysis method (HAM) technique as discussed by Khan et al [17] who used this method to solve a Fredholm integral equation system of the form \( u_i(x) = g_i(x) + \int_a^b K_i(x,s,u(s))\,ds, i = 1,2,\ldots,n \) analysed in terms of an equivalent system of non-linear operators such that \( F_i(u_1, u_2, \ldots, u_n) = 0, i = 1,2,\ldots,n \). In this work they report that the method can produce rapidly convergent successive approximations, however the main challenge with this technique is that the existence and convergence of the solutions are strongly dependent on the correct choice of parameters \( h, \Theta_1 \) and \( \Theta_2 \) which are used to develop and construct a series solution. Whilst a HAM technique may under certain circumstances offer the potential attraction of exact analytical solutions in special cases when the HAM reduces to the well known Adomain decomposition method (ADM), the overall benefits of homotopy based solution techniques is reduced in the present study. This is due to the extreme complexity that results in determining the optimal convergence-control parameter \( h \) for the large number of underlying simultaneous integral equations, and the additional challenges which occur when the HAM is applied to non-linear Fredholm equations.

As a result in this paper a numerical simulation approach has been developed which presents a conceptually appealing technique to determine the effective emissivity for a non-isothermal blackbody surface and which also allows for the incorporation of varying optical/thermal characteristics over the surface of the blackbody. The technique which is mathematically equivalent to a system of Fredholm integral equations also presents the advantages of avoiding indeterminate view factors and also allows for the inclusion of the zonal method or for the possibility of further mathematical refinement of the evaluation of the corresponding kernel contributions.

A potential area of future research that has been identified is the application of Markov Chain Monte Carlo (MCMC) techniques to computationally determine the exact von Neumann series solution of the underlying constituent equations as a type of hybrid deterministic/stochastic based technique.

**References**


