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In solar energy projects engineering designs make use of radiation configuration/view factors $F_{ij}$ in the initial concept, sizing and preliminary stages before full designs and analyses are performed with commercial software codes such as Ansys. Generally the view factors $F_{ij}$ are obtained from tables, charts, algebraic formulae or applications of the reciprocity principle. As a result the view factors are generally limited to variations of standard geometries utilizing the diffuse approximation.

In this paper/presentation we will examine the use of the Monte Carlo method as a means to directly compute the view factor and associated uncertainty quantification from geometrical uncertainties with application to concentrated solar power (CSP) designs.
Introduction

- Energy consumers in the public and private sectors are investigating alternative green energy resources such as plant material and animal waste biomass, algae, vegetable oil bio-fuels, wind energy and solar energy.
- In regions such as deserts where significant terrestrial solar irradiation is present solar farms from small/medium sizes from 7 MW to 32 MW to supplement the national electrical grid presents a potential cost-effective and efficient means for renewable green energy generation.
- Of various technical designs the use of a concentrated solar power (CSP) system consisting of an array of solar mirrors/lenses which focus the incident solar irradiation by reflection onto a central receiving point/zone such as an elevated tower is a technically attractive solution.
Introduction cont.

Figure: Example of a CSP system where an array of mirrors reflects the solar irradiation onto a central point/zone. Note that a subsurface of the mirror can be a quad element such an inclined rectangle and a subsurface of the tower can be a segment of a cylindrical surface.
In this paper:

1. We will examine the view factor $F_{ij}$ defined as the fraction of radiation leaving a surface $S_i$ such as a mirror and that is intercepted by a surface $S_j$ such as a central elevated tower which receives the reflected/focused solar radiation by means of a Monte Carlo simulations approach of the underlying mathematical model.

2. We will illustrate the computational statistical methodology to use a Monte Carlo simulation to compute the associated uncertainty quantification of the view factor $F_{ij}$ for specified inputs geometrical uncertainties.
Problem Geometry Orientation

Figure: Illustration of mathematical model of two interacting surfaces $S_i$ and $S_j$ where $S_i$ may be a portion of a mirror/lens and $S_j$ a portion of a focal point/zone.
Problem Mathematical Model

The view factor $F_{ij}$ for two interacting surfaces $S_i$ and $S_j$ where surface $S_i$ emits/reflects diffusively and the radiosity $J_i$ for surface $S_i$ is uniform is specified mathematically as

$$F_{ij} = \frac{1}{S_i} \int_{S_i} \int_{S_j} \frac{\cos \theta_i \cos \theta_j}{R^2} dS_i dS_j$$  \hspace{1cm} (1)

$$S_i F_{ij} = S_j F_{ji}$$  \hspace{1cm} (2)

Note that a surface $S_k \subset \mathbb{R}^3$ for some $k \in \mathbb{N}$ may be constructed and built up from a combination of surfaces viz.

$$S_i = \bigcup_{p=1}^{m} S_p$$  \hspace{1cm} (3)

$$S_j = \bigcup_{q=1}^{n} S_q$$  \hspace{1cm} (4)
Problem Mathematical Model cont.

The utility of decomposing a surface $S_k$ into a union of surfaces is that:

1. Grid discretizations for example meshes constructed from triangles and quadilaterals may be incorporated into the methodology demonstrated in this paper/presentation in a natural manner.

2. The surface integrals may be conveniently split into regions where radiation exchange is present and absent respectively.

3. The reciprocity principle $A_i F_{ij} = A_j F_{ji}$, and other relations such as $\sum_{j=1}^{n} F_{ij} = 1$, $F_{ij} = \sum_{k=1}^{m} F_{ik}$, $A_j F_{ji} = \sum_{k=1}^{m} A_k F_{ki}$, and $F_{ji} = (\sum_{k=1}^{m} A_k F_{ki}) / (\sum_{k=1}^{m} A_k)$ may be applied to simplify calculations.

The problem of analytically i.e. symbolically calculating the view between two arbitrary surfaces essentially reduces to calculating the view factor between two arbitrary subsurfaces which is only feasible in the event of simple regular geometries hence the need for numerical techniques.
Mathematical Framework

From the definition of a surface integral

\[ \int_S \phi(x, y, z) dS = \lim_{n \to \infty} \sum_{p=1}^{n} \phi(\xi_p, \eta_p, \zeta_p) \cdot \Delta S_p \]  

(5)

where \( \Delta S_p \) is an elemental area on the surface \( S \) for some point \( \vec{x}_p = (\xi_p, \eta_p, \zeta_p) \in A_p \) where \( A_p \) is the projection of \( \Delta S_p \) onto the \( xy \)-plane such that \( \Delta S_p = |\sec \gamma_p| \cdot \Delta A_p \) the surface integral may be evaluated as an area integral

\[ \int_S \phi(x, y, z) dS = \int_{\mathcal{R}} \phi(x, y, z) \cdot |\sec \gamma| dA \]  

(6)

\[ = \int_{\mathcal{R}} \phi(x, y, z) \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} dA \]  

(7)

where \( z = f(x, y) \) is an equation for the surface \( S(x, y, z) \), and the angle \( \gamma_p \) is calculated as \( |\sec \gamma_p| = 1/|\vec{n}_p \cdot \vec{k}| \) from vector geometry.
Mathematical Framework cont.

Using the above definition an arbitrary surface integral may be evaluated as an equivalent area integral as

$$\int_S \phi(x, y, z) dS = \int_{\mathcal{R}} \phi \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2} dydx$$

(8)

$$= \int_{\mathcal{R}} \phi(x, y, z) \frac{\sqrt{(\partial_x F)^2 + (\partial_y F)^2 + (\partial_z F)^2}}{|F_z|} dydx$$

(9)

where the surface is specified either explicitly as \( z = f(x, y) \) or implicitly as \( F(x, y, z) = 0 \). Similar equations can be used if the surface is projected onto the \( yz \)-plane of \( xz \)-plane.
Projection mappings $\Phi_i$ and $\Phi_j$

\[
\Phi_i : (x_i \in S_i \subset \mathbb{R}^3) \rightarrow (\xi_i \in \mathcal{R}_i \subset \mathbb{R}^2) \quad (10)
\]

\[
\Phi_j : (x_j \in S_j \subset \mathbb{R}^3) \rightarrow (\xi_j \in \mathcal{R}_j \subset \mathbb{R}^2) \quad (11)
\]

may be used to convert the surface integral into an equivalent area integral.

Let $\mathbf{r}_i = (x_i, y_i, z_i) \in S_i$ and $\mathbf{r}_j = (x_j, y_j, z_j) \in S_j$ so that with

\[
\mathbf{R}_{ij} = (x_j - x_i)i + (y_j - y_i)j + (z_j - z_i)k
\]

from standard vector geometry we have

\[
\frac{\cos \theta_i \cos \theta_j}{\pi R^2_{ij}} = \frac{\left(\mathbf{n}_i \cdot \mathbf{R}_{ij}\right)(-\mathbf{n}_j \cdot \mathbf{R}_{ij})}{\pi \|\mathbf{n}_i\| \cdot \|\mathbf{n}_j\| \cdot \|\mathbf{R}_{ij}\|^4} \quad (12)
\]

where $\mathbf{n}_i = \nabla F_i$ and $\mathbf{n}_j = \nabla F_j$
Mathematical Framework cont.

The application of a Monte Carlo integration to evaluate multidimensional integral $\int_V f dV$ for a finite dimensional domain $V \subset \mathbb{R}^n$ with $N$ Monte Carlo simulation events is

$$\int_V f dV = V \langle f \rangle \pm V \sqrt{\frac{\langle f^2 \rangle - \{\langle f \rangle\}^2}{N}} \quad (13)$$

$$\langle f \rangle = \frac{1}{N} \sum_{i=1}^{N} f(x_i) \quad (14)$$

$$\langle f^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} f^2(x_i) \quad (15)$$

In the application of a Monte Carlo integration in our view factor integral the Monte Carlo sampling will be over two dimensional regions $R_i$ and $R_j$ respectively.
Mathematical Framework cont.

Let the mapped regions $\mathcal{R}_i$ and $\mathcal{R}_j$ have boundaries $\Gamma_i$ and $\Gamma_j$ and ‘volumes’\(^1\) $\Omega_i$ and $\Omega_j$ respectively.

Applying the previous equations then yields\(^2\)

$$
\int_{S_i} \int_{S_j} \frac{\cos \theta_i \cos \theta_j}{\pi R_{ij}^2} = \sum_{q=1}^{N} \sum_{p=1}^{M} \left\{ \varphi(\xi_q, \xi_p) \right. \\
\times \left. \left[ \left( \frac{\Omega_i}{N} \right) \frac{\| \nabla F_i \|}{\| \nabla F_i \cdot k \|} \right] \left[ \left( \frac{\Omega_j}{M} \right) \frac{\| \nabla F_j \|}{\| \nabla F_j \cdot k \|} \right] \right\} \quad (16)
$$

where $\varphi(\xi_i, \xi_j) = (\cos \theta_i \cos \theta_j)/(\pi R_{ij}^2)$ and which demonstrates that a Monte Carlo integration of the mapped surface integrals will converge as $M \to \infty$ and $N \to \infty$ to the view factor $F_{ij}$ direct integration definition.

---

\(^1\)in a 2D space a volume is an area so $\Omega_i$ and $\Omega_j$ are simply the projected areas in the respective planes

\(^2\)Although this equation is for when both $S_i$ and $S_j$ are projected onto the $xy$-plane hence the presence of the vector $k$ any convenient plane projection may be used e.g. if $S_2$ is projected onto the $yz$-plane then use $i \implies \| \nabla F_j \cdot i \| = |\partial_x F_j|$
Mathematical Framework cont.

- A first principles direct numerical approach will converge to the view factor for a sufficiently large number of points.
- We have demonstrated that a Monte Carlo sampling will also converge to the view factor.
- The utility of this result is that one may now utilize a Monte Carlo sampling with random points in the respective domains which is less computationally expensive than a full direct numerical integration of all the points.
Practical Implementation of Monte Carlo Technique

- We have demonstrated the methodology for the general case where the surfaces $S_1$ and $S_2$ are algebraically expressed in terms of functions $f_1(x, y, z)$ and $f_2(x, y, z)$ however in a practical implementation there is a more straightforward approach that can be used.

- In practical configurations the surfaces may be specified in terms of grids which are discrete numerical data of elements $e_i$ with corresponding nodes $n_{j}^{(e_i)}$.

- For reasons that will be clear we opt for the use of quad elements instead of more traditional triangular elements that are typically used in CFD simulations.
Implementation Details

Consider a quad element as illustrated

![Figure: A three dimensional quad element](image)

\[
\begin{align*}
\mathbf{n}_1(e_i^{(S_1)}) &= [n_{1x}, n_{1y}, n_{1z}] \\
\mathbf{n}_2(e_i^{(S_1)}) &= [n_{2x}, n_{2y}, n_{2z}] \\
\mathbf{n}_3(e_i^{(S_1)}) &= [n_{3x}, n_{3y}, n_{3z}] \\
\mathbf{n}_4(e_i^{(S_1)}) &= [n_{4x}, n_{4y}, n_{4z}]
\end{align*}
\]
We calculate the centroid of the element $e_i^{(S_1)}$ as

$$c_{e_i^{(S_1)}} = \frac{1}{4} \sum_{k=1}^{4} n_k(e_i^{(S_1)})$$

(21)

and use this as a convenient choice from which the radiation from an element will be assumed to be emitted and received.
Implementation Details cont.

For a quad element the area is calculated using the cross product of the element edges as

$$A_{e_i^{(s_1)}} = \frac{1}{2} \| \mathbf{r}_{12} \times \mathbf{r}_{14} \| + \frac{1}{2} \| \mathbf{r}_{34} \times \mathbf{r}_{32} \|$$  (22)

where $\mathbf{r}_{12} = \mathbf{n}_2 - \mathbf{n}_1$, $\mathbf{r}_{14} = \mathbf{n}_4 - \mathbf{n}_1$, $\mathbf{r}_{34} = \mathbf{n}_4 - \mathbf{n}_3$, and $\mathbf{r}_{32} = \mathbf{n}_2 - \mathbf{n}_3$. The distance $R_{ij}$ between two elements $e_i^{(s_1)}$ and $e_j^{(s_2)}$ is calculated as the distance between the two elements centroids $\mathbf{c}_{e_i^{(s_1)}}$ and $\mathbf{c}_{e_j^{(s_2)}}$. In order to calculate the normals of the elements we directly compute the normal at the centroid as

$$\mathbf{N} = (\mathbf{n}_2 - \mathbf{c}) \times (\mathbf{n}_3 - \mathbf{c})$$  (23)

The area of the surface, say $S_1$, is simply calculated by the summation of the individual element areas as

$$A_{S_1} = \sum_{i=1}^{q} A_{e_i^{(s_1)}}$$  (24)
The view factor $F_{12}$ for surface $S_1$ and surface $S_2$ is then

$$F_{12} = \frac{1}{A_{S_1}} \sum_{i=1}^{q} \sum_{j=1}^{p} \left\{ \frac{\left[ N_{e_i}^{(S_1)} \cdot \left( c_{e_j}^{(S_2)} - c_{e_i}^{(S_1)} \right) \right] \left[ N_{e_j}^{(S_2)} \cdot \left( c_{e_i}^{(S_1)} - c_{e_j}^{(S_2)} \right) \right]}{\pi \left[ \left( c_{e_j}^{(S_2)} - c_{e_i}^{(S_1)} \right) \cdot \left( c_{e_j}^{(S_2)} - c_{e_i}^{(S_1)} \right) \right]^{2}} \times \frac{A_{e_i}^{(S_1)} A_{e_j}^{(S_2)}}{\left\| N_{e_i}^{(S_1)} \right\| \cdot \left\| N_{e_j}^{(S_2)} \right\| \cdot \left\| \left( c_{e_j}^{(S_2)} - c_{e_i}^{(S_1)} \right) \right\|^{2}} \right\}$$

$$= \frac{1}{A_{S_1}} \sum_{i=1}^{q} \sum_{j=1}^{p} \left[ N_{e_i}^{(S_1)} \cdot \left( c_{e_j}^{(S_2)} - c_{e_i}^{(S_1)} \right) \right] \left[ N_{e_j}^{(S_2)} \cdot \left( c_{e_i}^{(S_1)} - c_{e_j}^{(S_2)} \right) \right] \frac{A_{e_i}^{(S_1)} A_{e_j}^{(S_2)}}{\pi \left\| N_{e_i}^{(S_1)} \right\| \cdot \left\| N_{e_j}^{(S_2)} \right\| \cdot \left\| \left( c_{e_j}^{(S_2)} - c_{e_i}^{(S_1)} \right) \right\|^{4}}$$

Note that element areas $A_{e_i}^{(S_1)}$ is by definition the elemental area $\Delta S_q$ when the surface is made up of elements.
Implementation Details cont.

Assume that the data for two surfaces $S_1$ and $S_2$ have been loaded to files in which each of the surfaces have been discretized into $q$ elements $e_i^{(S_1)}$ ($i = 1, 2, \ldots, q$) for surface $S_1$ and $p$ elements $e_j^{(S_2)}$ ($j = 1, 2, \ldots, p$) for surface $S_2$ of form

$$
\begin{array}{cccccccccccc}
\mathbf{e}^{(S_1)} & n_1 & n_1 & n_1 & n_2 & n_2 & n_3 & n_3 & n_4 & n_4 & n_4 & n_4 \\
\mathbf{e}_1^{(S_1)} & A_1,1 & A_1,2 & A_1,3 & A_1,4 & A_1,5 & A_1,6 & A_1,7 & A_1,8 & A_1,9 & A_1,10 & A_1,11 & A_1,12 \\
\mathbf{e}_2^{(S_1)} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{e}_q^{(S_1)} & A_q,1 & A_q,2 & A_q,3 & A_q,4 & A_q,5 & A_q,6 & A_q,7 & A_q,8 & A_q,9 & A_q,10 & A_q,11 & A_q,12 \\
\end{array}
$$

so that $S_1$ is a $q \times 12$ matrix and similarly so that $S_2$ is a $p \times 12$ matrix. If all the elements were utilized then the previous formula would give the exact definition specification of the view factor: we have demonstrated that a Monte Carlo sampling will converge to this exact definition.
Implementation Strategy

- We know that for a sufficiently large number of sampling draws that the Monte Carlo technique will converge to the direct numerical integration definition.

- Instead of using an existing surface mathematical representation for example a quadratic surface fit and then sampling random points i.e. points $x_i \in S_i$ and $x_j \in S_j$ from these surfaces and computing the view factor $F_{ij}$ as $i \to \infty$ and $j \to \infty$ from these surfaces using projections $\Phi_i$ and $\Phi_j$ there might be an easier and numerically cheaper approach.

- Our strategy is simply to construct a very dense grid for $S_1$ and $S_2$ and instead of directly integrating the view factor from this grid to simply randomly sample element areas from the existing grid.

- The grid has the elements already numbered viz. $e_1^{(s_1)}, \ldots, e_q^{(s_1)}$ for surface $S_1$ and similarly for surface $S_2$ so to randomly sample points reduces to simply randomly sampling element numbers $1, \ldots, q$ since the node coordinates are already associated to the element number.
Implementation Strategy cont.

- To randomly sample integer points in a sequence $1, \ldots, q$ for surface $S_1$ elements adopt the following algorithm
  \[ Z = k(q - 1) + 1 \]  
  where $k \sim U(0, 1)$ is a number that follows a standard uniform distribution i.e. $k$ is a random number such that $0 \leq k \leq 1$ with a rectangular probability distribution
- Look up the corresponding element $e_{kS_1}$
- Use the same approach and randomly sample from the element numbers $1, \ldots, p$ for surface $S_2$ elements with some other random number $t \sim U(0, 1)$ say for a corresponding random integer $W$ say
- We now have two random elements $e_{Z(S_1)}$ and $e_{W(S_2)}$ which after perturbation consistent to the geometrical uncertainty can be added to the calculation for $F_{ij}$
- Repeat the integer sampling process $M$ times for $S_1$ and $N$ times for $S_2$ and each time with the sampled elements add the respective contribution to $F_{ij}$
- Work out the final $F_{ij}$ from these MC simulation events & post-process to compute the PDF’s for the UQ
Implementation Strategy cont.

**Simulation 1**

- \( S_1 \)
- \( F_{12}(\text{sim #1}) \)

**Simulation 2**

- \( F_{12}(\text{sim #2}) \)

**Simulation 100000**

- \( F_{12}(\text{sim #100000}) \)

**Figure:** Schematic illustration of implementation strategy
Simulations

We perform numerical simulations for two surfaces where surface $S_1$ is a rectangle inclined at an angle and surface $S_2$ is a portion of a cylinder for a specified angular arc that is offset. The corresponding equations are:

$$S_1 := \left\{(x_1, y_1, z_1) | x_1 = H \cos \alpha, y_1 = W, \right.$$ 
$$\left. z_1 = H \sin \alpha, 0 \leq H \leq H_{\text{max}}, 0 \leq y \leq W \right\} \quad (29)$$

$$S_2 := \left\{(x_2, y_2, z_2) | x_2 = \rho \cos \theta + x_o, y_2 = \rho \sin \theta + y_o, \right.$$ 
$$\left. z_2 = z + z_o, \theta_{\text{min}} \leq \theta \leq \theta_{\text{max}}, z_{\text{min}} \leq z \leq z_{\text{max}} \right\} \quad (30)$$
Numerical Simulations cont.

Figure: Schematic illustration of surface $S_1$
Numerical Simulations cont.

Figure: Schematic illustration of surface $S_2$
Numerical Simulations cont.

Figure: Schematic illustration of surface $S_1$ and surface $S_2$
Numerical Simulations cont.

Figure: Illustration of results from a Monte Carlo simulation with 250 simulation events for the inclined rectangular plate $S_1$ and the portion of the offset elevated cylinder $S_2$ with the surface geometrical uncertainties assumed as Gaussian probability density functions.
Discussion

- We have demonstrated the convergence and equivalence between the Monte Carlo method and direct integration approaches in the calculation of the view factor for two finite surfaces.
- Based on this equivalence we have proposed that the computational cost may be reduced by random sampling in the surface domains when building up the view factor.
- Utilizing the Monte Carlo sampling then leads to a natural mechanism to incorporate the underlying geometrical uncertainties in the respective surfaces.
- The geometrical uncertainties are incorporated into the view factor calculations by simultaneous sampling from the associated probability density functions such as Gaussian distributions.
- The final results may be post-processed in a straightforward manner to yield simultaneous knowledge of the view factor $F_{ij}$ and its standard/expanded uncertainty $u(F_{ij})$ for specified coverage factors.

*End of Presentation / Appendix Overleaf*
Appendix - Validation of Computer Code

The numerical algorithm was tested with a code written in MATLAB for the geometry of perpendicular rectangles with a common edge as illustrated below.

Surface \( S_i \) had a width of \( X \) and a depth of \( Y \) and surface \( S_j \) had a width of \( X \) and a height of \( Z \) such that

\[
S_1 = \{(x_1, y_1, z_1)|0 \leq x_1 \leq X, 0 \leq y_1 \leq Y, z = 0\} \quad (31)
\]
\[
S_2 = \{(x_2, y_2, z_2)|0 \leq x_2 \leq X, y_2 = Y, 0 \leq z_2 \leq Z\} \quad (32)
\]
This simple geometry has a rather complicated view factor analytical formula with $H = Z/X$ and $W = Y/X$ as

$$F_{ij} = \frac{1}{\pi W} \left( W \arctan \frac{1}{W} + H \arctan \frac{1}{H} \right)$$

$$- \left( H^2 + W^2 \right)^{1/2} \arctan \frac{1}{\left( H^2 + W^2 \right)^{1/2}}$$

$$+ \frac{1}{4} \ln \left\{ \frac{(1 + W^2)(1 + H^2)}{1 + W^2 + H^2} \left[ \frac{W^2(1 + W^2 + H^2)}{(1 + W^2)(W^2 + H^2)} \right]^{W^2} \right. $$

$$\times \left[ \frac{H^2(1 + H^2 + W^2)}{(1 + H^2)(H^2 + W^2)} \right]^{H^2} \right\} \right)$$

(33)
Appendix - Validation of Computer Code cont.

Test case: $X = 1.00$, $Y = 2.00$, $Z = 3.00$ with a grid of $20 \times 20$ for $S_1$ and $20 \times 20$ for $S_2$ yields the following results:

$$F_{12}^{(Monte Carlo)} = 0.165286867804961 \quad (34)$$

$$F_{12}^{(Analytical)} = 0.161694014333028 \quad (35)$$

therefore the error $e_{(MC)/(DI)}$ between the view factor results from the Monte Carlo simulation and the direct integration is approximately

$$e_{(MC)/(DI)} = 2.22\% \quad (36)$$

In practice a minimum of $10^3$ sampled points will be used instead of $(20 - 1) \times (20 - 1) = 361$ elements in surfaces $S_1$ and $S_2$. Recall that we opt for the use of quad elements instead of triangular elements to reduce the algebraic complexity in the mapping

$$\Phi_k : (x_k \in S_k) \rightarrow (\xi_k \in R_k), \ k = i, j$$

since triangular elements cannot be uniquely mapped.
For the validation test case of two perpendicular rectangles with a common edge the view factor geometry was also coded in MATHEMATICA where the symbolic results confirmed the MATLAB numerical results.

This also very clearly demonstrated that a purely symbolic implementation for the double iterated view factor surface integral is impractical for practical engineering problems.

A direct comparison between the GUM and Monte Carlo UQ frameworks is only possible where a measurand function adequately satisfies the following conditions:

1. The non-linearity must be insignificant.
2. The Central Limit Theorem (CLT) from statistics must be valid for the model.
3. The conditions for the Welch-Satterthwaite formula must be satisfied in the effective degrees of freedom calculations.

These criteria are not satisfied for view factor computations hence the need for alternatives such as Monte Carlo simulations which is the focus of the research in this proceeding/presentation.
Mathematica code page 1 of 9

ClearAll["Global`*"]

\[ F_1 = \frac{1}{\pi W} \left( \frac{W}{\pi} \arctan \left( \frac{1}{W} \right) + \frac{1}{2} \arctan \left( \frac{1}{H^2 + W^2} \right) \right) + \]
\[ \frac{1}{4} \log \left[ \frac{1 + W^2}{1 + W^2 + H^2} \left( \frac{W^2 + (1 + W^2 + H^2)^2}{(1 + W^2)^2} \right) \right] \]
\[ \frac{1}{\pi W} \left( \frac{H}{H} \arctan \left( \frac{1}{H} \right) + \frac{W}{W} \arctan \left( \frac{1}{W} \right) \right) + \frac{1}{4} \log \left[ \frac{1 + H^2}{1 + H^2 + W^2} \left( \frac{H^2 + (1 + H^2 + W^2)^2}{(1 + H^2)^2} \right) \right] \]

\[ F_2 = F_1 / H \rightarrow \frac{Z}{X} \]
\[ \frac{1}{\pi W} \left( \frac{W}{W} \arctan \left( \frac{1}{W} \right) + \frac{Z}{Z} \arctan \left( \frac{1}{Z} \right) \right) + \frac{1}{4} \log \left[ \frac{1 + W^2}{1 + W^2 + Z^2} \left( \frac{W^2 + (1 + W^2 + Z^2)^2}{(1 + W^2)^2} \right) \right] \]

\[ F_3 = F_2 / W \rightarrow \frac{Y}{X} \]
\[ \frac{1}{\pi Y} \left( \frac{Y}{Y} \arctan \left( \frac{1}{Y} \right) + \frac{Z}{Z} \arctan \left( \frac{1}{Z} \right) \right) + \frac{1}{4} \log \left[ \frac{1 + Y^2}{1 + Y^2 + Z^2} \left( \frac{Y^2 + (1 + Y^2 + Z^2)^2}{(1 + Y^2)^2} \right) \right] \]
\[ F = F_3 \]

\[
\frac{1}{\pi Y} \left( \frac{Y \arctan \left( \frac{X}{Y} \right) + Z \arctan \left( \frac{X}{Z} \right)}{X} - \sqrt{\frac{Y^2 + Z^2}{X^2}} \arctan \left[ \frac{1}{\sqrt{\frac{Y^2 + Z^2}{X^2}}} \right] \right) + \frac{1}{4} \log \left[ \frac{1 + \frac{Y^2}{X^2}}{1 + \frac{Z^2}{X^2}} \right] + \frac{1}{4} \log \left[ \frac{1 + \frac{Y^2}{X^2}}{1 + \frac{Z^2}{X^2}} \right] \left( \frac{2 \left( 1 + \frac{Y^2}{X^2} \right) \left( \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} \right)} \right) \]

\[ F_{12}[X_, Y_, Z_] = F \]

\[
\frac{1}{\pi Y} \left( \frac{Y \arctan \left( \frac{X}{Y} \right) + Z \arctan \left( \frac{X}{Z} \right)}{X} - \sqrt{\frac{Y^2 + Z^2}{X^2}} \arctan \left[ \frac{1}{\sqrt{\frac{Y^2 + Z^2}{X^2}}} \right] \right) + \frac{1}{4} \log \left[ \frac{1 + \frac{Y^2}{X^2}}{1 + \frac{Z^2}{X^2}} \right] + \frac{1}{4} \log \left[ \frac{1 + \frac{Y^2}{X^2}}{1 + \frac{Z^2}{X^2}} \right] \left( \frac{2 \left( 1 + \frac{Y^2}{X^2} \right) \left( \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} \right)} \right) \]

\[ C_x = D[F_{12}[X, Y, Z], X] \]
\[
\frac{1}{\pi Y X} \left( \frac{1}{Y \left(1 + \frac{X^2}{Y^2}\right)} + \frac{1}{X \left(1 + \frac{Y^2}{X^2}\right)} + \frac{-2 \frac{Y^2}{X^2} - 2 \frac{Z^2}{X^2}}{2 \left(\frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)} \left(1 + \frac{1}{\frac{X}{Y} + \frac{Y}{X}}\right) - \frac{Y \text{ArcTan} \left[\frac{X}{Y}\right]}{X^2} - \frac{Z \text{ArcTan} \left[\frac{X}{Z}\right]}{X^2} \right)
\]

\[
\frac{2 \sqrt{\frac{Y^2}{X^2} + \frac{Z^2}{X^2}}}{\text{ArcTan} \left[\frac{1}{\sqrt{\frac{Y^2}{X^2} + \frac{Z^2}{X^2}}\right] + 1/ \left(\frac{1 + \frac{Y^2}{X^2} \left(1 + \frac{Z^2}{X^2}\right)}{\frac{Y^2}{X^2} \left(1 + \frac{Z^2}{X^2}\right)} + \frac{1 + \frac{Y^2}{X^2} \left(1 + \frac{Z^2}{X^2}\right)}{\frac{Y^2}{X^2} \left(1 + \frac{Z^2}{X^2}\right)} - \frac{Y^2 \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)}{X^2 \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)} \right) \right) \left(1 + \frac{Y^2 + Z^2}{X^2} \right) -
\]

\[
\left(\frac{2 \left(1 + \frac{Y^2}{X^2}\right) \frac{X^2}{Y^2} \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)}{X^2 \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)} \right) \left(\frac{Z^2 \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)}{X^2 \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)} \right) \right) \left(1 + \frac{Y^2 + Z^2}{X^2} \right) \right) \right) -
\]

\[
\left(2 \frac{Y^2 \left(1 + \frac{Z^2}{X^2}\right)}{X^2 \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)} \right) \left(\frac{Z^2 \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)}{X^2 \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)} \right) \right) \left(1 + \frac{Y^2 + Z^2}{X^2} \right) +
\]

\[
\frac{1}{1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}} \left(\frac{1 + \frac{Y^2}{X^2}}{X^2 \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)} \right) \left(\frac{Z^2 \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)}{X^2 \left(1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2}\right)} \right) \right) \left(1 + \frac{Y^2 + Z^2}{X^2} \right) \right) \right) -
\]
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\[
\frac{1}{\pi Y} \left( Y \text{ArcTan} \left[ \frac{X}{Y} \right] + Z \text{ArcTan} \left[ \frac{X}{Z} \right] - \sqrt{\frac{Y^2 + Z^2}{X^2}} \text{ArcTan} \left[ \frac{1}{\sqrt{\frac{Y^2}{X^2} + \frac{Z^2}{X^2}}} \right] \right) + \frac{1}{4} \log \left( \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)
\]
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\[
\frac{1}{\pi Y} \left( - \frac{1}{1 + \frac{Y^2}{X^2}} \right) + \frac{Y}{X^2 \left( \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right) \left( 1 + \frac{1}{\frac{Z}{X}} \right) + \frac{1}{\pi Y} \left( - \frac{1}{1 + \frac{Y^2}{X^2}} \right) \right) + \frac{Y \text{ArcTan} \left[ \frac{Y}{X} \right]}{X^2 \left( \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} + \frac{Y \text{ArcTan} \left[ \frac{Y}{X} \right]}{X^2 \left( \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}
\]

\[
\frac{1}{4 \left( 1 + \frac{Y^2}{X^2} \right)} \left( 1 + \frac{Z^2}{X^2} \right) \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right) \left( \frac{Y^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{1}{2}} \left( \frac{Z^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{1}{2}}
\]

\[
\left( 1 + \frac{Y^2}{X^2} \right) \left( 1 + \frac{Z^2}{X^2} \right) \left( \frac{Y^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{1}{2}} \left( \frac{Z^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{1}{2}} \left( \frac{2 Y Z^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{1}{2}}
\]

\[
\left( 1 + \frac{1}{\frac{Z}{X}} \right) Z^2 \left( 1 + \frac{Z^2}{X^2} \right) \left( \frac{Y^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{1}{2}} \left( \frac{Z^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{1}{2}} \left( \frac{2 Y Z^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{1}{2}}
\]

\[
C_Y = D[F_{12}[X, Y, Z], X]
\]

\[
1 \left( 1 + \frac{Y^2}{X^2} \right) \left( 1 + \frac{Z^2}{X^2} \right) \left( \frac{Y^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{1}{2}} \left( \frac{Z^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{1}{2}}
\]
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\[ \left( x^2 \left( \frac{1 + y^2 + z^2}{x^2 + y^2} \right) + \frac{1}{1 + \frac{y^2}{x^2} + \frac{z^2}{x^2}} \right) \left( 1 + \frac{y^2}{x^2} \right) \left( 1 + \frac{z^2}{x^2} \right) \left( \frac{y^2}{x^2} \left( 1 + \frac{y^2}{x^2} + \frac{z^2}{x^2} \right) \right) \left( \frac{z^2}{x^2} \left( 1 + \frac{y^2}{x^2} + \frac{z^2}{x^2} \right) \right) \]

\[ \frac{1}{x^2 \left( 1 + \frac{y^2}{x^2} \right) \left( 1 + \frac{z^2}{x^2} \right)} \left( \frac{y^2}{x^2} \left( 1 + \frac{y^2}{x^2} + \frac{z^2}{x^2} \right) \right) \frac{2 \left( 1 + \frac{y^2}{x^2} + \frac{z^2}{x^2} \right)}{x^2 \left( 1 + \frac{y^2}{x^2} \right) \left( 1 + \frac{z^2}{x^2} \right)} - \frac{2 \left( 1 + \frac{y^2}{x^2} \right) \left( 1 + \frac{z^2}{x^2} \right)}{x^2 \left( 1 + \frac{y^2}{x^2} + \frac{z^2}{x^2} \right)} - 2 \log \left( 1 + \frac{y^2}{x^2} \right) \left( 1 + \frac{z^2}{x^2} \right) \left( 1 + \frac{y^2}{x^2} + \frac{z^2}{x^2} \right) \frac{\sqrt{\frac{y^2}{x^2} + \frac{z^2}{x^2}}}{Y \sqrt{\frac{Y^2}{x^2} + \frac{Z^2}{x^2}}} + \frac{Z \sqrt{\frac{Y^2}{x^2} + \frac{Z^2}{x^2}}}{2} \]
In[91] = F
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\[
\frac{1}{\pi Y} \left( \frac{Y \arctan \left( \frac{X}{Y} \right) + Z \arctan \left( \frac{X}{Z} \right)}{X} - \sqrt{\frac{Y^2}{X^2} + \frac{Z^2}{X^2}} \arctan \left( \frac{1}{\sqrt{\frac{Y^2}{X^2} + \frac{Z^2}{X^2}}} \right) \right) + \\
\frac{1}{4} \log \left[ \frac{1}{\left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right) \left( 1 + \frac{X^2}{Y^2} \right) \left( 1 + \frac{X^2}{Z^2} \right)} \left( \frac{Y^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{x^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{y^2}{x^2}} \left( \frac{Z^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{x^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{z^2}{x^2}} \right] \]
\]

\text{In[98]} = \text{ViewFactor} = F_{12}[X, Y, Z]

\text{Out[98]} = \frac{1}{\pi Y} \left( \frac{Y \arctan \left( \frac{X}{Y} \right) + Z \arctan \left( \frac{X}{Z} \right)}{X} - \sqrt{\frac{Y^2}{X^2} + \frac{Z^2}{X^2}} \arctan \left( \frac{1}{\sqrt{\frac{Y^2}{X^2} + \frac{Z^2}{X^2}}} \right) \right) + \\
\frac{1}{4} \log \left[ \frac{1}{\left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right) \left( 1 + \frac{X^2}{Y^2} \right) \left( 1 + \frac{X^2}{Z^2} \right)} \left( \frac{Y^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{x^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{y^2}{x^2}} \left( \frac{Z^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{x^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right)^{\frac{z^2}{x^2}} \right] \]

\text{In[100]} = \text{ViewFactorTest1.nb}

\text{Out[100]} = \frac{1}{2 \pi} \left( 3 \arctan \left( \frac{1}{3} \right) + 2 \arctan \left( \frac{1}{2} \right) - \sqrt{13} \arctan \left( \frac{1}{\sqrt{13}} \right) + \frac{1}{4} \log \left( 21964383255760327839744 \right) \right)
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\[
\frac{1}{\pi Y} \left( \frac{Y \text{ArcTan} \left( \frac{X}{Y} \right)}{X} + \frac{Z \text{ArcTan} \left( \frac{X}{Z} \right)}{X} - \sqrt{\frac{Y^2 + Z^2}{X^2}} \text{ArcTan} \left[ \frac{1}{\sqrt{\frac{Y^2}{X^2} + \frac{Z^2}{X^2}}} \right] + \frac{1}{4} \log \left[ \frac{1}{\left( 1 + \frac{Y^2}{X^2} \right) \left( 1 + \frac{Z^2}{X^2} \right) \left( 1 + \frac{Z^2}{X^2} \right)} \left( \frac{Y^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Y^2}{X^2} \right) \left( \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right) \left( \frac{Z^2 \left( 1 + \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)}{X^2 \left( 1 + \frac{Z^2}{X^2} \right) \left( \frac{Y^2}{X^2} + \frac{Z^2}{X^2} \right)} \right) \right] \right] \right)
\]

\text{ViewFactorNumerical} = N[\text{ViewFactor}]

\text{Out[101]} = 0.161694